T_i-spaces, I

Fatma Bayoumi

Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt

Ismail Ibedou

Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt

Abstract

In this and a subsequent paper, using the notion of fuzzy neighborhood filter which has been defined in [5], we introduce and study the separation axioms T_i, $i = 0, 1, 2, 3, 4$ in the case of a fuzzy topology. These axioms are related only to usual points and ordinary subsets. In the classical case $L = \{0, 1\}$ these axioms are the usual ones. These axioms fulfills many properties analogous to the usual axioms. Whereas this paper is devoted to the axioms $T_0,\ T_1,\ T_2$, in part II the axioms $T_3,\ T_4$ are introduced and studied.

Keywords: Fuzzy filters, Principal fuzzy filters, Fuzzy neighborhood filters, Valued fuzzy neighborhoods, Fuzzy topologies, Fuzzy separation axioms.

Introduction

There are many definitions for the fuzzy separation axioms depend on fuzzy points [9, 10]. In this paper we introduce fuzzy separation axioms using only the usual points and the ordinary subsets. A notion related to usual points, called *fuzzy neighborhood filter* at a point, is used to define these axioms.

The notion of *fuzzy filter* has been introduced by Eklund and Gähler in [2]. By means of an extension of this notion of fuzzy filter, a point-based approach to fuzzy topology related to usual points has been developed by Gähler in [4,5]. In this approach several notions are related to usual points, between these notions the notion of fuzzy neighbourhood filter which is defined by means of the notion of interior of a fuzzy set. For each fuzzy topological space, the mapping which assigns to each point x the fuzzy neighborhood filter at x can be considered itself as the fuzzy topology.

These fuzzy separation axioms depends only on usual points and so the work with these axioms will be more simple and more general. We study here the cases $i = 0, 1, 2$. These fuzzy separation axioms are good extensions in sense of Lowen [12], this means an induced fuzzy topological space $(X, \omega(T))$ is T_i if and only if the underlying topological space (X, T) is T_i. The implications between these axioms goes well, that is, each T_i-space is T_{i-1} for $i = 1, 2$. Moreover, for each fuzzy topological space (X, τ) which is T_i, the α-level topological space (X, τ_α), $\alpha \in L_1$ and the initial topological space $(X, \iota(\tau))$ are T_i. We also show that the initial and final fuzzy topological spaces of a family of T_i-spaces, $i = 0, 1, 2$, are also T_i-spaces. Therefore the fuzzy topological product space, subspace, sum space and quotient space of T_i-spaces, $i = 0, 1, 2$, are also T_i.

Gähler defined in [6] and [7] separation axioms for the convergence space using the convergence $\mathcal{M} \to x$ of an element $\mathcal{M}$ of ϕX to an element x of X.

These axioms in the special case of a fuzzy topology are equivalent to our axioms. This specialization we obtain in replacing the convergence $\mathcal{M} \to x$ by $\mathcal{M} \leq \mathcal{N}(x)$ where $\mathcal{N}(x)$ is the fuzzy neighborhood filter at x.

1. On Fuzzy Neighborhood Filters

Throughout the paper let L be a complete chain with different least and last elements 0 and 1, respectively. Let $L_0 = L \setminus \{0\}$ and $L_1 = L \setminus \{1\}$.

By a *fuzzy subset* of a set X we mean a mapping $f : X \to L$ of X into L. Denote by L^X and $P(X)$ for the sets of all fuzzy subsets and of all ordinary subsets of X, respectively. For each $x \in X$ and $\alpha \in L_0$, the fuzzy subset x_α of X whose value α at x and 0 otherwise is called the *fuzzy point* in X. For each $\alpha \in L$, the constant fuzzy subset of X with value α will be denoted by $\overline{\alpha}$.

A *fuzzy topology* on a set X ([1,8]) is a subset τ of L^X which contains the constant fuzzy sets $\overline{0}$ and $\overline{1}$ and is closed with respect to finite infima and arbitrary suprema. The pair (X, τ) is called a fuzzy topological space and the elements of τ are called open fuzzy sets. The *interior* $\mathrm{int}_\tau f$ of a fuzzy set f is the greatest open fuzzy set less than or equal to f, that is,

$$\mathrm{int}_\tau f \;=\; \bigvee_{g \in \tau,\, g \leq f} g. \tag{1.1}$$

For each fuzzy set $f \in L^X$, the strong α-cut and the weak α-cut of f are the subsets $s_\alpha f = \{x \in X \mid f(x) > \alpha\}$ and $w_\alpha f = \{x \in X \mid f(x) \geq \alpha\}$ of X, respectively.

If τ_1 and τ_2 are fuzzy topologies on X, then τ_1 is said to be *finer than* τ_2, denoted by, $\tau_1 \leq \tau_2$, provided $\tau_1 \supseteq \tau_2$. For each fuzzy topology τ on X, the α-*level* and the *initial* topologies of τ are defined by: $\tau_\alpha = \{s_\alpha f \mid f \in \tau\}$ and $\iota(\tau) = \inf\{\tau_\alpha \mid \alpha \in L_1\}$ respectively, where inf is the infimum with respect to the finer relation on fuzzy topologies. If T is an ordinary topology on X, then the *induced fuzzy topology* on X is given by.

$$\omega(T) \;=\; \{f \in L^X \mid s_\alpha f \in T \text{ for all } \alpha \in L_1\}.$$

Initial and final fuzzy topological spaces. Consider a family of fuzzy topological spaces $((X_i, \tau_i))_{i \in I}$ and for each $i \in I$ a mapping $f_i : X \to X_i$. By the *initial fuzzy topology* of $(\tau_i)_{i \in I}$ with respect to $(f_i)_{i \in I}$ is meant the coarsest fuzzy topology τ on X for which all mappings $f_i : (X, \tau) \to (X_i, \tau_i)$ are fuzzy continuous. τ is defined as in ([11]) by the supremum of the family $(f_i^{-1}(\tau_i))_{i \in I}$ with respect to the finer relation on fuzzy topologies, that is, $\tau = \bigvee\limits_{i \in I} f_i^{-1}(\tau_i)$, where $f_i^{-1}(\tau_i) = \{f_i^{-1}(g) \mid g \in \tau_i\}$. It is easily seen that $f_i^{-1}(\tau_i)$ is the initial fuzzy topology of τ_i with respect to f_i.

Let X be the cartesian product $\prod\limits_{i \in I} X_i$ of the family $(X_i)_{i \in I}$ and $p_i : X \to X_i$ be the related projections. $(X, \tau = \bigvee_{i \in I} p_i^{-1}(\tau_i))$ is called the *fuzzy topological product space* of the family $((X_i, \tau_i))_{i \in I}$. It is clear that $\tau = \bigvee_{i \in I} p_i^{-1}(\tau_i)$ is the initial fuzzy topology of $(\tau_i)_{i \in I}$ with respect to $(p_i)_{i \in I}$.

eIf (X, τ) is a fuzzy topological space, A is a non-empty subset of X, and $i : A \hookrightarrow X$ is the inclusion mapping, then $(A, \tau_A = i^{-1}(\tau))$ is called the *fuzzy topological subspace* of (X, τ). τ_A is the initial fuzzy topology of τ with respect to i.

Assume now that $f_i : X_i \to X$ is a mapping of X_i into X. By the *final fuzzy topology* of $(\tau_i)_{i \in I}$ with respect to $(f_i)_{i \in I}$ we mean the finest fuzzy topology τ on X for which all mappings $f_i : (X_i, \tau_i) \to (X, \tau)$ are fuzzy continuous. τ is defined in [11] as the infimum of the family $(f_i(\tau_i))_{i \in I}$ with respect to the finer relation on fuzzy topologies, that is, $\tau = \bigwedge_{i \in I} f_i(\tau_i)$, where $f_i(\tau_i) = \{\lambda \in L^X \mid f_i^{-1}(\lambda) \in \tau_i\}$ is the final fuzzy topology of τ_i with respect to f_i.

Let $X = \biguplus_{i \in I} X_i = \bigcup_{i \in I} (X_i \times \{i\})$ be the disjoint union of the family $(X_i)_{i \in I}$ and $e_i : X_i \to X$, and for each $i \in I$ the related canonical injections are defined by $e_i(x_i) = (x_i, i)$. Then $(X, \bigwedge_{i \in I} e_i(\tau_i))$ is called the *fuzzy topological sum space* of the family $((X_i, \tau_i))_{i \in I}$. $\tau = \bigwedge_{i \in I} e_i(\tau_i)$ is the final fuzzy topology of $(\tau_i)_{i \in I}$ with respect to $(e_i)_{i \in I}$.

If (X, τ) is a fuzzy topological space and $f : X \to Y$ a surjective mapping, then $(Y, f(\tau))$ is called the *fuzzy topological quotient space*.

Fuzzy open and fuzzy closed mappings. Let (X, τ) and (Y, σ) be fuzzy topological spaces. The mapping $f : (X, \tau) \to (Y, \sigma)$ is called *fuzzy open* (*fuzzy closed*) if the image $f(g)$ of the open (closed) fuzzy sets g with respect to τ is open (closed) with respect to σ.

Fuzzy filters. Let X be a non-empty set. By a *fuzzy filter* on X ([2,4]) is meant a mapping $\mathcal{M} : L^X \to L$ such that the following conditions are fulfilled.

(F1) $\mathcal{M}(\overline{\alpha}) \leq \alpha$ holds for all $\alpha \in L$ and $\mathcal{M}(\overline{1}) = 1$.

(F2) $\mathcal{M}(f \wedge g) = \mathcal{M}(f) \wedge \mathcal{M}(g)$ for all $f, g \in L^X$.

A fuzzy filter $\mathcal{M}$ is called *homogeneous* if $\mathcal{M}(\overline{\alpha}) = \alpha$ for all $\alpha \in L$.

If $\mathcal{M}$ and $\mathcal{N}$ are fuzzy filters on X, $\mathcal{M}$ is said to be *finer than* $\mathcal{N}$, denoted by, $\mathcal{M} \leq \mathcal{N}$, provided $\mathcal{M}(f) \geq \mathcal{N}(f)$ holds for all $f \in L^X$. By $\mathcal{M} \not\leq \mathcal{N}$ we denote that $\mathcal{M}$ is not finer than $\mathcal{N}$. If L is a complete chain, then

$$\mathcal{M} \not\leq \mathcal{N} \iff \text{ there is } f \in L^X \text{ such that } \mathcal{M}(f) < \mathcal{N}(f).$$

For each fuzzy filter $\mathcal{M}$ on X, the subset $\alpha\text{-pr}\,\mathcal{M}$ of L^X defined by:

$$\alpha\text{-pr}\,\mathcal{M} \;=\; \{f \in L^X \mid \mathcal{M}(f) \geq \alpha\}$$

is a prefilter on X, where a non-empty subset $\mathcal{F}$ of L^X is called a *prefilter* on X ([13]) if the following conditions are fulfilled.

(P1) $\overline{0} \notin \mathcal{F}$.

(P2) $f, g \in \mathcal{F}$ implies $f \wedge g \in \mathcal{F}$.

(P3) $f \in \mathcal{F}$ and $f \leq g$ imply $g \in \mathcal{F}$.

Proposition 1.1 [4] *Let A be a set of fuzzy filters on X. Then the following are equivalent.*

(1) *The infimum $\bigwedge_{\mathcal{M} \in A} \mathcal{M}$ of A with respect to the finer relation of fuzzy filters exists.*

(2) *For each non-empty finite subset $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$ of A we have $\mathcal{M}_1(f_1) \wedge \cdots \wedge \mathcal{M}_n(f_n) \leq \sup(f_1 \wedge \cdots \wedge f_n)$ for all $f_1, \ldots, f_n \in L^X$.*

(3) *For each $\alpha \in L_0$ and each non-empty finite subset $f_1, \ldots, f_n$ of $\bigcup_{\mathcal{M} \in A} \alpha\text{-pr}\,\mathcal{M}$ we have $\alpha \leq \sup(f_1 \wedge \cdots \wedge f_n)$.*

Ultra fuzzy filters. A fuzzy filter $\mathcal{M}$ on X is called an *ultra* fuzzy filter if it does not have a properly finer fuzzy filter.

Proposition 1.2 [2,4] *For each fuzzy filter on a set X, there exists a finer ultra fuzzy filter.*

Fuzzy neighborhood filters. For each fuzzy topological space (X, τ) and each $x \in X$ the mapping $\mathcal{N}(x) : L^X \to L$ defined by

$$\mathcal{N}(x)(f) = \text{int}_\tau f(x) \tag{1.2}$$

for all $f \in L^X$ is a fuzzy filter on X, called the *fuzzy neighborhood filter* of the space (X, τ) at x ([5]).

For each $x \in X$, the mapping $\dot{x} : L^X \to L$ defined by $\dot{x}(f) = f(x)$ for all $f \in L^X$ is a homogeneous fuzzy filter on X.

The fuzzy neighborhood filters fulfill the following conditions.

(N1) $\dot{x} \le \mathcal{N}(x)$ holds for all $x \in X$.

(N2) $\mathcal{N}(x)(y \mapsto \mathcal{N}(y)(f)) = \mathcal{N}(x)(f)$ for all $x \in X$ and $f \in L^X$.

Note that the mapping $y \mapsto \mathcal{N}(y)(f)$ is the fuzzy set $\text{int}_\tau f$.

Proposition 1.3 [5] *There is a one-to-one correspondence between the fuzzy topologies τ and the mappings $x \mapsto \mathcal{N}(x)$ of X into the set of all fuzzy filters on X, where for each $x \in X$, $\mathcal{N}(x)$ fulfills* (N1) *and* (N2)*. This correspondence is given by $\mathcal{N}(x)(f) = \text{int}_\tau f(x)$.*

Valued fuzzy neighborhoods. Let (X, τ) be a fuzzy topological space. Then for each $\alpha \in L_0$, the fuzzy subsets $f \in L^X$ are called *α-fuzzy neighborhoods* at x ([5]) if $f \in \alpha\text{-pr}\,\mathcal{N}(x)$, that is,

$$\alpha \le \text{int}_\tau f(x). \tag{1.3}$$

By a *valued fuzzy neighborhood* at x ([5]) is meant an α-fuzzy neighborhood at x for some $\alpha \in L_0$.

The closure operator. For each fuzzy set $f \in L^X$, the fuzzy set $\mathrm{cl}_\tau f \in L^X$ defined by

$$\mathrm{cl}_\tau f(x) = \bigvee_{\mathcal{M} \leq \mathcal{N}(x)} \mathcal{M}(f) \tag{1.4}$$

for all $x \in X$, is called the *closure* of f ([5]). It is easily seen that $\mathrm{cl}_\tau f \geq f$.

For each fuzzy topological space (X, τ) the closure operator of τ is the mapping cl which assigns to each fuzzy filter $\mathcal{M}$ on X the fuzzy filter $\mathrm{cl}\,\mathcal{M}$, where

$$\mathrm{cl}\,\mathcal{M}(f) = \bigvee_{\mathrm{cl}_\tau g \leq f} \mathcal{M}(g). \tag{1.5}$$

$\mathrm{cl}\,\mathcal{M}$ is called the *closure* of $\mathcal{M}$. cl is isotone and is a hull operator, that is, for all fuzzy filters $\mathcal{M}$ and $\mathcal{N}$ on X, we have

$$\mathcal{M} \leq \mathcal{N} \text{ implies } \mathrm{cl}\,\mathcal{M} \leq \mathrm{cl}\,\mathcal{N} \tag{1.6}$$

and moreover cl fulfills that

$$\mathcal{M} \leq \mathrm{cl}\,\mathcal{M} \tag{1.7}$$

holds ([3]).

2. T_0-Spaces

This section is devoted to introduce a notion of T_0-spaces in the fuzzy case using the neighborhood filter at a point. We also introduce different equivalent definitions and study its relation with the α-level and the initial topologies and also we show that this notion is a good extension in sense of Lowen [12]. Moreover, we show that the initial and final fuzzy topological spaces of T_0-spaces are also T_0.

Definition 2.1 A fuzzy topological space (X, τ) is called T_0 if for all $x, y \in X$ with $x \neq y$ we have $\dot{x} \not\leq \mathcal{N}(y)$ or $\dot{y} \not\leq \mathcal{N}(x)$.

In the classical case $L = \{0, 1\}$ the filter $\mathcal{N}(x)$ is up to an identification a set of neighborhoods of x and the filter $\dot{x}$ is a set of subsets of X contain x. In this case

$\dot{x} \leq \mathcal{N}(y)$ means $\dot{x} \supseteq \mathcal{N}(y)$ and $\dot{x} \not\leq \mathcal{N}(y)$ means there is a neighborhood of y not containing x.

The following results will be used in the proof of Theorem 2.1.

Lemma 2.1 *For all $x, y \in X$ with $x \neq y$ we have*

$$\mathcal{N}(x) \not\leq \mathcal{N}(y) \text{ implies } \dot{x} \not\leq \mathcal{N}(y).$$

Proof. Since L is a complete chain, then $\mathcal{N}(x) \not\leq \mathcal{N}(y)$ means that there is $f \in L^X$ such that $\mathcal{N}(x)(f) < \mathcal{N}(y)(f)$. From that $\mathrm{int}_\tau \mathrm{int}_\tau f = \mathrm{int}_\tau f$ and $\mathcal{N}(x)(f) = \mathrm{int}_\tau f(x)$ it follows $\dot{x}(\mathrm{int}_\tau f) < \mathcal{N}(y)(\mathrm{int}_\tau f)$. Thus there is a $g = \mathrm{int}_\tau f \in L^X$ such that $\dot{x}(g) < \mathcal{N}(y)(g)$. Hence, $\dot{x} \not\leq \mathcal{N}(y)$. $\square$

Lemma 2.2 *Let $\mathcal{L}, \mathcal{M}, \mathcal{N}$ be fuzzy filters on a set X. Then we have:*

(1) $\mathcal{L} \not\leq \mathcal{M} \geq \mathcal{N}$ *implies* $\mathcal{L} \not\leq \mathcal{N}$.

(2) $\mathcal{L} \geq \mathcal{M} \not\leq \mathcal{N}$ *implies* $\mathcal{L} \not\leq \mathcal{N}$.

Proof. Since $\mathcal{L} \not\leq \mathcal{M} \geq \mathcal{N}$ and $\mathcal{N} \geq \mathcal{L}$ give a contradiction, then $\mathcal{L} \not\leq \mathcal{M} \geq \mathcal{N}$ implies $\mathcal{L} \not\leq \mathcal{N}$. That is, (1) is fulfilled.

The proof of (2) goes similarly. $\square$

Lemma 2.3 *For all $x, y \in X$ with $x \neq y$ the following statements are equivalent.*

(1) $\mathrm{cl}\,\dot{x} \not\leq \dot{y}$ *or* $\mathrm{cl}\,\dot{y} \not\leq \dot{x}$.

(2) $\dot{x} \not\leq \mathrm{cl}\,\dot{y}$ *or* $\dot{y} \not\leq \mathrm{cl}\,\dot{x}$.

(3) $\mathrm{cl}\,\dot{x} \neq \mathrm{cl}\,\dot{y}$.

Proof. Direct. $\square$

In the following theorem there will be introduced some equivalent definitions for the T_0-spaces.

Theorem 2.1 *Let (X,τ) be a fuzzy topological space. Then the following statements are equivalent.*

(1) *(X,τ) is T_0.*

(2) *For all $x,y\in X$ with $x\neq y$ we have $\mathrm{cl}\,\dot{x}\not\leq\dot{y}$ or $\mathrm{cl}\,\dot{y}\not\leq\dot{x}$.*

(3) *$x\neq y$ implies $\mathcal{N}(x)\neq\mathcal{N}(y)$ for all $x,y\in X$.*

(4) *$x\neq y$ implies there is $f\in L^X$ such that $\alpha\leq\mathcal{N}(x)(f)$ and $f(y)<\alpha$ for some $\alpha\in L_0$ or there is $g\in L^X$ such that $\beta\leq\mathcal{N}(y)(g)$ and $g(x)<\beta$ for some $\beta\in L_0$ for all $x,y\in X$.*

Proof. (1) $\Rightarrow$ (2): Let (1) be hold and let $x\neq y$ in X. Then $\dot{x}\not\leq\mathcal{N}(y)$ or $\dot{y}\not\leq\mathcal{N}(x)$ and since $\mathrm{cl}\,\dot{x}\geq\dot{x}$ and $\dot{x}\leq\mathcal{N}(x)$ for each $x\in X$, it follows by (1) and (2) in Lemma 2.2 that $\mathrm{cl}\,\dot{x}\not\leq\dot{y}$ or $\mathrm{cl}\,\dot{y}\not\leq\dot{x}$. That is, (2) is fulfilled.

(2) $\Rightarrow$ (3): For all $x\neq y$ in X, from Lemma 2.3 we have $\mathrm{cl}\,\dot{x}\not\leq\dot{y}$ implies $\dot{x}\not\leq\mathrm{cl}\,\dot{y}$ which implies $\mathcal{N}(x)\geq\dot{x}\not\leq\mathrm{cl}\,\dot{y}\geq\dot{y}$. So, $\mathcal{N}(x)\not\leq\dot{y}$ and then for $g=\mathrm{int}_\tau f\in L^X$, we get $\mathcal{N}(x)(g)<g(y)$. i.e. $\mathrm{int}_\tau\,\mathrm{int}_\tau f(x)=\mathrm{int}_\tau f(x)<\mathrm{int}_\tau f(y)$. That is, $\mathcal{N}(x)\not\leq\mathcal{N}(y)$. Hence, $\mathcal{N}(x)\neq\mathcal{N}(y)$ and therefore (3) is fulfilled.

(3) $\Rightarrow$ (4): If (3) is fulfilled and $x,y\in X$ with $x\neq y$, then $\mathcal{N}(x)\neq\mathcal{N}(y)$, that is, $\mathcal{N}(x)\not\leq\mathcal{N}(y)$ or $\mathcal{N}(y)\not\leq\mathcal{N}(x)$. Lemma 2.1 implies that $\dot{x}\not\leq\mathcal{N}(y)$ or $\dot{y}\not\leq\mathcal{N}(x)$. Thus there is a fuzzy set $f\in L^X$ such that $f(x)<\mathcal{N}(y)(f)$ or a fuzzy set $g\in L^X$ such that $g(y)<\mathcal{N}(x)(g)$. If we take $\mathcal{N}(y)(f)=\alpha$ and $\mathcal{N}(x)(g)=\beta$, then we get $\alpha\leq\mathcal{N}(y)(f)$ and $f(x)<\alpha$ or $\beta\leq\mathcal{N}(x)(g)$ and $g(y)<\beta$. Hence, (4) holds.

(4) $\Rightarrow$ (1): Now, let (4) be hold and let $x\neq y$. Then there is a fuzzy set $f\in L^X$ such that $\dot{y}(f)=f(y)<\mathcal{N}(x)(f)$ or a fuzzy set $g\in L^X$ such that $\dot{x}(g)=g(x)<\mathcal{N}(y)(g)$. This is equivalent to $\dot{x}\not\leq\mathcal{N}(y)$ or $\dot{y}\not\leq\mathcal{N}(x)$ and hence, (1) is fulfilled. □

In the view of Lemma 2.3 the condition (2) in Theorem 2.1 can be written as:

(2′) $x\neq y$ implies $\dot{x}\not\leq\mathrm{cl}\,\dot{y}$ or $\dot{y}\not\leq\mathrm{cl}\,\dot{x}$;

or

(2″) $x \neq y$ implies $\mathrm{cl}\,\dot{x} \neq \mathrm{cl}\,\dot{y}$.

Example 2.1 Let L be a complete chain, $X = \{1,2\}$ and let $\tau = \{\overline{0}, \overline{1}\}$ be the indiscrete fuzzy topology. Since $\mathrm{int}_\tau f = \overline{0}$ or $\overline{1}$ for all $f \in L^X$, then for $x = 1$ and $y = 2$ we have

$$\mathcal{N}(1)(f) = \mathrm{int}_\tau f(1) = \begin{cases} 1 & if\ f = \overline{1} \\ 0 & if\ f \neq \overline{1} \end{cases} = \mathrm{int}_\tau f(2) = \mathcal{N}(2)(f)$$

for all $f \in L^X$ and hence $\mathcal{N}(1) = \mathcal{N}(2)$. That is, the space (X, τ) is not T_0.

A subset of a space (X, T) is called a neighborhood of a point $x \in X$, denoted by $\mathcal{O}_x$, if there is an open set $G \in T$ such that $x \in G \subseteq \mathcal{O}_x$.

A topological space (X, T) is called T_0 if $x \neq y$ implies there is a neighborhood $\mathcal{O}_x$ of x such that $y \notin \mathcal{O}_x$ or there is a neighborhood $\mathcal{O}_y$ of y such that $x \notin \mathcal{O}_y$.

The next proposition shows that the fuzzy separation axiom T_0 is a good extension in sense of [12].

Proposition 2.1 *A topological space (X, T) is T_0 if and only if the induced fuzzy topological space $(X, \omega(T))$ is T_0.*

Proof. Let (X, T) be T_0 and let $x \neq y$. Then there is a neighborhood $\mathcal{O}_y \in T$ such that $x \notin \mathcal{O}_y$. Taking $f = \chi_{\mathcal{O}_y}$, since $s_\alpha f = \mathcal{O}_y$ for each $\alpha \in L_1$ it follows $\mathcal{N}(y)(f) = (\mathrm{int}_{\omega(T)} f)(y) = f(y) = 1 > f(x)$. Hence $\dot{x} \not\leq \mathcal{N}(y)$, where $\mathcal{N}(y)$ is the fuzzy neighborhood filter at y related to the fuzzy topology $\omega(T)$, that is, $(X, \omega(T))$ is T_0.

Now, let $(X, \omega(T))$ be T_0. Then $x \neq y$ implies there is a $f \in L^X$ such that $f(x) < (\mathrm{int}_{\omega(T)} f)(y)$. Since $\mathrm{int}_{\omega(T)} f \in \omega(T)$ and $(\mathrm{int}_{\omega(T)} f)(x) \leq f(x)$ it follows $y \in s_{f(x)}(\mathrm{int}_{\omega(T)} f) \in T$ and $x \notin s_{f(x)}(\mathrm{int}_{\omega(T)} f)$. Hence $s_{f(x)}(\mathrm{int}_{\omega(T)} f)$ is a neighborhood of y not containing x and therefore (X, T) is T_0. $\square$

Remark 2.1 If S_1 and S_2 are topologies on a set X and S_2 is finer than S_1, then

$$(X, S_1) \text{ is } T_i \text{ implies } (X, S_2) \text{ is } T_i \text{ for } i = 0, 1, 2.$$

Proposition 2.2 *Let (X, τ) be a fuzzy topological space and let (X, τ_α) and $(X, \iota(\tau))$ be the α-level and the initial topological spaces of (X, τ), respectively. Then the following statements*

(1) *(X, τ) is T_0;*

(2) *(X, τ_α) is T_0, $\alpha \in L_1$;*

(3) *$(X, \iota(\tau))$ is T_0*

fulfill the following implications: (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2): Let (X, τ) be T_0 and let $x \neq y$ in X. Then there is $f \in L^X$ such that $f(x) < (\text{int}_\tau f)(y)$ and hence $y \in s_{f(x)}(\text{int}_\tau f)$ and $x \notin s_{f(x)}(\text{int}_\tau f)$. Since $\text{int}_\tau f \in \tau$, it follows $s_{f(x)}(\text{int}_\tau f) \in \tau_{f(x)}$ and thus $s_{f(x)}(\text{int}_\tau f)$ is a neighborhood of y not containing x. Hence the space (X, τ_α), for $\alpha \leq f(x)$, is T_0.

(2) $\Rightarrow$ (3): Since $\iota(\tau)$ is finer than τ_α for all $\alpha \in L_1$ and the space (X, τ_α) is T_0 for some $\alpha \in L_1$, then Remark 2.1 implies that the space $(X, \iota(\tau))$ is T_0. $\square$

In the following we shall show that if I is a class and for each $i \in I$, (X_i, τ_i) is a T_0-space and for some $i \in I$, $f_i : X \to X_i$ is an injective mapping and τ is the initial fuzzy topology of $(\tau_i)_{i \in I}$ with respect to $(f_i)_{i \in I}$, then the initial fuzzy topological space (X, τ) is also T_0.

At first we shall consider the case of I being a singleton.

Proposition 2.3 *Let (Y, σ) be a T_0-space and let $f : X \to Y$ be an injective mapping. Then the initial fuzzy topological space $(X, f^{-1}(\sigma))$ is also T_0.*

Proof. f is injective means $x \neq y$ in X implies $f(x) \neq f(y)$ in Y and (Y, σ) is T_0-space means there exists $g \in L^Y$ such that

$$g(f(x)) < \alpha \leq (\mathrm{int}_\sigma g)(f(y))$$

for some $\alpha \in L_0$. Since $f : (X, f^{-1}(\sigma)) \to (Y, \sigma)$ is fuzzy continuous, then we have $(\mathrm{int}_\sigma g) \circ f \leq \mathrm{int}_{f^{-1}(\sigma)}(g \circ f)$ and thus

$$(g \circ f)(x) < \alpha \leq (\mathrm{int}_{f^{-1}(\sigma)}(g \circ f))(y).$$

This means there exists $h = g \circ f \in L^X$ and $\alpha \in L_0$ such that

$$h(x) < \alpha \leq (\mathrm{int}_{f^{-1}(\sigma)} h)(y).$$

Hence $(X, f^{-1}(\sigma))$ is T_0-space. $\Box$

Now consider the case of any class I.

Proposition 2.4 *Let (X_i, τ_i) be a T_0-space for all $i \in I$ and let $f_i : X \to X_i$ be an injective mapping for some $i \in I$. Then the initial fuzzy topological space (X, τ) is also T_0.*

Proof. Let $x \neq y$ in X. Since f_i is an injective for some $i \in I$, then $f_i(x) \neq f_i(y)$ in X_i and thus there exists $\lambda_i \in L^{X_i}$ and $\alpha_i \in L_0$ such that

$$\lambda_i(f_i(x)) < \alpha_i \leq (\mathrm{int}_{\tau_i} \lambda_i)(f_i(y)).$$

Because of that $f_i : (X, \tau) \to (X_i, \tau_i)$ is fuzzy continuous, then $(\mathrm{int}_{\tau_i} \lambda_i) \circ f_i \leq \mathrm{int}_\tau(\lambda_i \circ f_i)$ and therefore

$$(\lambda_i \circ f_i)(x) < \alpha_i \leq \mathrm{int}_\tau(\lambda_i \circ f_i)(y).$$

Therefore there exists $\lambda = \lambda_i \circ f_i \in L^X$ and $\alpha_i \in L_0$ such that the condition of T_0-space is fulfilled. Hence, (X, τ) is T_0-space. $\Box$

Since the fuzzy topological subspace and product space are special initial fuzzy topological spaces, then we have the following result.

Corollary 2.1 Propositions 2.3 and 2.4 imply that the fuzzy topological subspace and the fuzzy topological product space of T_0-spaces are also T_0.

Now let for all $i \in I$, (X_i, τ_i) is a T_0-space and for some $i \in I$, $f_i : X_i \to X$ is a surjective fuzzy open mapping and τ is the final fuzzy topology of $(\tau_i)_{i \in I}$ with respect to $(f_i)_{i \in I}$. The following propositions show that the final fuzzy topological space (X, τ) is also T_0.

Proposition 2.5 *Let (X, τ) be a T_0-space and let $f : X \to Y$ be a surjective fuzzy open mapping. Then the final fuzzy topological space $(Y, f(\tau))$ is also T_0.*

Proof. Since f is surjective, then $y_1 \neq y_2$ in Y implies there are $x_1, x_2 \in X$ such that $y_1 = f(x_1)$, $y_2 = f(x_2)$ and $x_1 \neq x_2$. From that (X, τ) is a T_0-space it follows there are $g \in L^X$ and $\alpha \in L_0$ such that

$$g(x_1) < \alpha \leq (\mathrm{int}_\tau g)(x_2)$$

and this means

$$g(f^{-1}(y_1)) < \alpha \leq (\mathrm{int}_\tau g)(f^{-1}(y_2))$$

which means

$$(f(g))(y_1) < \alpha \leq (f(\mathrm{int}_\tau g))(y_2).$$

Because of that f is fuzzy open, it follows $f(\mathrm{int}_\tau g) \leq \mathrm{int}_{f(\tau)}(f(g))$ and therefore

$$(f(g))(y_1) < \alpha \leq (\mathrm{int}_{f(\tau)} f(g))(y_2).$$

Since $f(g) \in L^Y$, then we get that the final fuzzy topological space $(Y, f(\tau))$ is T_0.
□

Proposition 2.6 *Let I be any class and (X_i, τ_i) be a T_0-space for all $i \in I$ and $f_i : X_i \to X$ be a surjective fuzzy open mapping for some $i \in I$. Then the final fuzzy topological space (X, τ) is also T_0.*

Proof. Let $x \neq y$ in X. From that f_i is surjective it follows there are $x_i, y_i \in X_i$ such that $x = f_i(x_i)$, $y = f_i(y_i)$ and $x_i \neq y_i$. Since (X_i, τ_i) is a T_0-space, then there are $\lambda_i \in L^{X_i}$ and $\alpha_i \in L_0$ such that

$$\lambda_i(x_i) < \alpha_i \leq (\mathrm{int}_{\tau_i}\lambda_i)(y_i)$$

which is equivalent to

$$\lambda_i(f_i^{-1}(x)) < \alpha_i \leq (\mathrm{int}_{\tau_i}\lambda_i)(f_i^{-1}(y))$$

and this means

$$(f_i(\lambda_i))(x) < \alpha_i \leq (f_i(\mathrm{int}_{\tau_i}\lambda_i))(y),$$

f_i is fuzzy open implies $f_i(\mathrm{int}_{\tau_i}\lambda_i) \leq \mathrm{int}_{\tau}(f_i(\lambda_i))$ and this implies

$$(f_i(\lambda_i))(x) < \alpha_i \leq (\mathrm{int}_{\tau} f_i(\lambda_i))(y)$$

for some $f_i(\lambda_i) \in L^X$ and some $\alpha_i \in L_0$. Hence the final fuzzy topological space (X, τ) is T_0. $\square$

The following result is a direct consequence of Propositions 2.5 and 2.6.

Corollary 2.2 The fuzzy topological sum space and the fuzzy topological quotient space of T_0-spaces are also T_0.

In the following it will be shown that the finer fuzzy topological space of T_0-space is also T_0.

Proposition 2.7 *Let (X, τ) be a T_0-space and let σ be a fuzzy topology on X finer than τ. Then (X, σ) is also T_0-space.*

Proof. Since σ is finer than τ, then $\mathrm{int}_{\tau} f \leq \mathrm{int}_{\sigma} f$ for all $f \in L^X$. From that (X, τ) is T_0-space it follows for all $x \neq y$ in X there exists $f \in L^X$ and $\alpha \in L_0$ such that

$$f(x) < \alpha \leq (\mathrm{int}_{\tau} f)(y)$$

and thus

$$f(x) < \alpha \leq (\mathrm{int}_{\sigma} f)(y).$$

Hence (X, σ) is also T_0-space. $\square$

3. T_1-Spaces

Here the fuzzy separation axioms T_1 spaces will be introduced and a similar study for T_0-spaces will be done for T_1-spaces.

Definition 3.1 A fuzzy topological space (X, τ) is called T_1 if for all $x, y \in X$ with $x \neq y$ we have $\dot{x} \nleq \mathcal{N}(y)$ and $\dot{y} \nleq \mathcal{N}(x)$.

Proposition 3.1 *Every T_1-space is T_0-space.*

Proof. Obvious. □

The following example shows that there are T_0-spaces which are not T_1-spaces.

Example 3.1 Let L be a complete chain, $X = \{x, y\}$ and let $\tau = \{\overline{0}, \overline{1}, x_1\}$. Then the fuzzy topological space (X, τ) is T_0 and not T_1.

The following theorem introduces equivalent definitions for the T_1-spaces.

Theorem 3.1 *In a fuzzy topological space (X, τ) the following statements are equivalent.*

(1) (X, τ) *is* T_1.

(2) $x \neq y$ *implies* $\operatorname{cl} \dot{x} \nleq \dot{y}$ *and* $\operatorname{cl} \dot{y} \nleq \dot{x}$ *for all* $x, y \in X$.

(3) $\operatorname{cl} \dot{x} = \dot{x}$ *for each* $x \in X$.

(4) $x \neq y$ *implies* $\bigvee\limits_{f \in \mathcal{V}_x, f \neq \overline{1}} f = g$ *is a* 1 *- fuzzy neighborhood at* x *and* $g(y) < 1$, *where* $\mathcal{V}_x = \{f \in L^X \mid f$ *is a valued fuzzy neighborhood at* $x\}$.

(5) $x \neq y$ *implies there are* $f, g \in L^X$ *such that* $\alpha \leq \mathcal{N}(x)(f)$, $f(y) < \alpha$ *and* $\beta \leq \mathcal{N}(y)(g)$, $g(x) < \beta$ *for some* $\alpha, \beta \in L_0$.

Proof. (1) $\Rightarrow$ (2): Similarly as in (1) $\Rightarrow$ (2) in Theorem 2.1.

(2) $\Rightarrow$ (3): Let (2) be hold. Then $\mathrm{cl}\,\dot{x} \leq \dot{y}$ or $\mathrm{cl}\,\dot{y} \leq \dot{x}$ implies $x = y$. This means $\mathrm{cl}\,\dot{x} \leq \dot{x}$ and we have $\mathrm{cl}\,\dot{x} \geq \dot{x}$ for each $x \in X$. Hence, $\mathrm{cl}\,\dot{x} = \dot{x}$.

(3) $\Rightarrow$ (4): Let $\mathrm{cl}\,\dot{x} = \dot{x}$, $\mathcal{V}_x = \{f \in L^X \mid f$ is a valued fuzzy neighborhood at $x\}$ for each $x \in X$ and let $x \neq y$. Then $\bigvee\limits_{\mathrm{cl} g \leq h} g(x) = h(x)$ and

$$\mathrm{int}(\bigvee_{f\in\mathcal{V}_x,\, f\neq\overline{1}} f)(x) \geq \bigvee_{f\in\mathcal{V}_x,\, f\neq\overline{1}} \mathrm{int} f(x) \;=\; \mathrm{cl} f(x)$$

for each $x \in X$. Hence, $\bigvee\limits_{f\in\mathcal{V}_x,\, f\neq\overline{1}} f = g$ is $1-$ fuzzy neighborhood at x and $g(y) = \bigvee\limits_{f\in\mathcal{V}_x,\, f\neq\overline{1}} f(y) < 1$ for all $y \neq x$. That is, (4) holds.

(4) $\Rightarrow$ (5): If $\mathrm{int}(\bigvee\limits_{f\in\mathcal{V}_x,\, f\neq\overline{1}} f)(x) = \mathrm{int} g(x) = 1$ and $g(y) < 1$ for all $y \neq x$, then taking $\alpha = 1$, we get $\alpha \leq \mathcal{N}(x)(g)$ and $g(y) < \alpha$ for some $\alpha \in L_0$. Similarly, we get that there is $k \in L^X$ such that $\beta \leq \mathcal{N}(y)(k)$ and $k(x) < \beta$ for some $\beta \in L_0$. Hence, (5) holds.

(5) $\Rightarrow$ (1): Let (5) be hold and $x \neq y$. Then there are $f, g \in L^X$ such that

$$f(y) < \alpha \leq \mathcal{N}(x)(f) \text{ and } g(x) < \beta \leq \mathcal{N}(y)(g)$$

for some $\alpha, \beta \in L_0$. Hence, $\dot{y} \nleq \mathcal{N}(x)$ and $\dot{x} \nleq \mathcal{N}(y)$ and thus (1) is fulfilled. $\square$

Example 3.2 Let L be a complete chain, $X = \{x, y\}$ and let $\tau = \{\overline{0}, \overline{1}, x_1, y_1\}$. Then for $x \neq y$ there are $f = x_1$ and $g = y_1$ such that

$$f(y) = 0 < 1 = \mathrm{int}_\tau f(x) = \mathcal{N}(x)(f) \;\text{ and }\; g(x) = 0 < 1 = \mathrm{int}_\tau g(y) = \mathcal{N}(y)(g)$$

that is, $\dot{y} \nleq \mathcal{N}(x)$ and $\dot{x} \nleq \mathcal{N}(y)$. Hence, (X, τ) is T_1.

A topological space (X, T) is called T_1 if $x \neq y$ implies there are neighborhoods $\mathcal{O}_x$ and $\mathcal{O}_y$ of x and y, respectively such that $y \notin \mathcal{O}_x$ and $x \notin \mathcal{O}_y$.

Proposition 3.2 *A topological space (X, T) is T_1 if and only if the induced fuzzy topological space $(X, \omega(T))$ is T_1.*

Proof. Similarly, as in Proposition 2.1. $\square$

Proposition 3.3 *For every fuzzy topological space* (X, τ) *the following statements*

(1) (X, τ) *is* T_1;

(2) (X, τ_α) *is* T_1, $\alpha \in L_1$;

(3) $(X, \iota(\tau))$ *is* T_1

fulfill the following implications: (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. As in Proposition 2.2. $\square$

The following propositions will show that the initial fuzzy topological space (X, τ) of the family $((X_i, \tau_i))_{i \in I}$ of T_1-spaces is also T_1.

Consider the case of one mapping.

Proposition 3.4 *Let* (Y, σ) *be a* T_1*-space and let* $f : X \to Y$ *be an injective mapping. Then the initial fuzzy topological space* $(X, f^{-1}(\sigma))$ *is also* T_1.

Proof. Let $x \neq y$ in X. Since f is injective, then $f(x) \neq f(y)$ in Y and thus (Y, σ) is T_1-space means there exist $g, h \in L^Y$ such that

$$g(f(x)) < \alpha \leq (\mathrm{int}_\sigma g)(f(y)) \text{ and } h(f(y)) < \beta \leq (\mathrm{int}_\sigma h)(f(x))$$

for some $\alpha,\ \beta \in L_0$. From that $f : (X, f^{-1}(\sigma)) \to (Y, \sigma)$ is fuzzy continuous it follows $(\mathrm{int}_\sigma g) \circ f \leq \mathrm{int}_{f^{-1}(\sigma)}(g \circ f)$ for all $g \in L^Y$ and thus

$$(g \circ f)(x) < \alpha \leq (\mathrm{int}_{f^{-1}(\sigma)}(g \circ f))(y) \text{ and } (h \circ f)(y) < \beta \leq (\mathrm{int}_{f^{-1}(\sigma)}(h \circ f))(x).$$

This means there exist $k = g \circ f, l = h \circ f \in L^X$ and $\alpha,\ \beta \in L_0$ such that

$$k(x) < \alpha \leq (\mathrm{int}_{f^{-1}(\sigma)} k)(y) \text{ and } l(y) < \beta \leq (\mathrm{int}_{f^{-1}(\sigma)} l)(x).$$

Hence $(X, f^{-1}(\sigma))$ is T_1-space. $\square$

Now consider the case of any class I.

Proposition 3.5 *Let (X_i, τ_i) be a T_1-space for all $i \in I$ and let $f_i : X \to X_i$ be an injective mapping for some $i \in I$. Then the initial fuzzy topological space (X, τ) is also T_1.*

Proof. The proof goes similarly as in the case of T_0. $\square$

The following result is a direct consequence of Propositions 3.4 and 3.5.

Corollary 3.1 The fuzzy topological subspace and the fuzzy topological product space of T_1-spaces are also T_1.

Now we are going to show that the final fuzzy topological space (X, τ) of a family $((X_i, \tau_i))_{i \in I}$ of T_1-spaces is also T_1.

Proposition 3.6 *Let (X, τ) be a T_1-space and let $f : X \to Y$ be a surjective fuzzy open mapping. Then the final fuzzy topological space $(Y, f(\tau))$ is also T_1.*

Proof. Since f is surjective, then $y_1 \neq y_2$ in Y implies there are $x_1, x_2 \in X$ such that $y_1 = f(x_1)$, $y_2 = f(x_2)$ and $x_1 \neq x_2$. Thus there are $g, h \in L^X$ and $\alpha,\ \beta \in L_0$ such that

$$g(x_1) < \alpha \leq (\mathrm{int}_\tau g)(x_2) \text{ and } h(x_2) < \beta \leq (\mathrm{int}_\tau h)(x_1)$$

and this means

$$g(f^{-1}(y_1)) < \alpha \leq (\mathrm{int}_\tau g)(f^{-1}(y_2)) \text{ and } h(f^{-1}(y_2)) < \beta \leq (\mathrm{int}_\tau h)(f^{-1}(y_1))$$

which means

$$(f(g))(y_1) < \alpha \leq (f(\mathrm{int}_\tau g))(y_2) \text{ and } (f(h))(y_2) < \beta \leq (f(\mathrm{int}_\tau h))(y_1).$$

Because of that f is fuzzy open, it follows $f(\mathrm{int}_\tau g) \leq \mathrm{int}_{f(\tau)}(f(g))$ for all $g \in L^X$ and therefore

$$(f(g))(y_1) < \alpha \leq (\mathrm{int}_{f(\tau)} f(g))(y_2) \text{ and } (f(h))(y_2) < \beta \leq (\mathrm{int}_{f(\tau)} f(h))(y_1).$$

Since $f(g), f(h) \in L^Y$, then we get that the final fuzzy topological space $(Y, f(\tau))$ is T_1. $\square$

Proposition 3.7 *Let I be any class and (X_i, τ_i) be a T_1-space for all $i \in I$ and $f_i : X_i \to X$ be a surjective fuzzy open mapping for some $i \in I$. Then the final fuzzy topological space (X, τ) is also T_1.*

Proof. It is similar to the proof of Proposition 2.6. $\Box$

The following result is a direct consequence of Propositions 3.6 and 3.7.

Corollary 3.2 The fuzzy topological sum space and the fuzzy topological quotient space of T_1-spaces are also T_1.

Now, we shall show that the finer fuzzy topological space of T_1-space is also T_1.

Proposition 3.8 *Let (X, τ) be a T_1-space and let σ be a fuzzy topology on X finer than τ. Then (X, σ) is also T_1-space.*

Proof. Similarly as in the case of T_0-space. $\Box$

4. T_2-Spaces

Here, using the neighborhood filter introduced in [5], we introduce and study the Hausdorff notion in the fuzzy case.

Definition 4.1 A fuzzy topological space (X, τ) is called T_2 or *Hausdorff* if for all $x, y \in X$ with $x \neq y$ we have $\mathcal{N}(x) \wedge \mathcal{N}(y)$ does not exist.

Proposition 4.1 *Every T_2-space is T_1-space.*

Proof. Let (X, τ) be a T_2-space and $x \neq y$. Then $\mathcal{N}(x) \wedge \mathcal{N}(y)$ does not exist and this means there are $f, g \in L^X$ such that $\mathcal{N}(x)(f) \wedge \mathcal{N}(y)(g) > \sup(f \wedge g)$. Thus

$$\mathcal{N}(x)(f) > (f \wedge g)(y) \text{ and } \mathcal{N}(y)(g) > (f \wedge g)(x). \tag{4.1}$$

From condition $(F2)$ of the fuzzy filter and (4.1) we get

$$\mathcal{N}(x)(f \wedge g) > (f \wedge g)(y) \text{ and } \mathcal{N}(y)(f \wedge g) > (f \wedge g)(x),$$

that is, there are $k = f \wedge g \in L^X$ such that $\mathcal{N}(x)(k) > k(y)$ and $\mathcal{N}(y)(k) > k(x)$ and this means $\dot{y} \not\leq \mathcal{N}(x)$ and $\dot{x} \not\leq \mathcal{N}(y)$. Hence, (X, τ) is T_1. $\square$

The class of T_1-spaces is larger than the class of T_2-spaces. This will be shown in the following example.

Example 4.1 Let L be a complete chain, $X = \{x, y\}$ and $\tau = \{x_\alpha, y_\alpha, \overline{\alpha}$ for each $\alpha \in L\}$. Then there are $f = x_1$ and $g = y_1$ such that

$$\text{int}_\tau f(x) = 1 > f(y) \text{ and } \text{int}_\tau g(y) = 1 > g(x).$$

Hence, (X, τ) is T_1 but it is not T_2.

Theorem 4.1 *For a fuzzy topological space (X, τ), the following statements are equivalent.*

(1) *(X, τ) is T_2.*

(2) *For all $x, y \in X$ with $x \neq y$, we have $\mathcal{M} \not\leq \mathcal{N}(x)$ or $\mathcal{M} \not\leq \mathcal{N}(y)$ for all ultra fuzzy filters $\mathcal{M}$ on X.*

(3) *For all $x, y \in X$ with $x \neq y$, we have $\mathcal{M} \not\leq \mathcal{N}(x)$ or $\mathcal{M} \not\leq \mathcal{N}(y)$ for all fuzzy filters $\mathcal{M}$ on X.*

Proof. (1) $\Rightarrow$ (2): If (1) is fulfilled and $x \neq y$, $\mathcal{M} \leq \mathcal{N}(x)$ and $\mathcal{M} \leq \mathcal{N}(y)$ for all ultra fuzzy filters $\mathcal{M}$ on X, then for all $f, g \in L^X$ we get

$$\mathcal{M}(f) \geq \mathcal{N}(x)(f) \text{ and } \mathcal{M}(g) \geq \mathcal{N}(y)(g).$$

Since $\mathcal{M}(f) \leq \sup f$ for each $f \in L^X$ and $\mathcal{M}(f \wedge g) = \mathcal{M}(f) \wedge \mathcal{M}(g)$ it follows

$$\mathcal{N}(x)(f) \wedge \mathcal{N}(y)(g) \leq \mathcal{M}(f \wedge g) \leq \sup (f \wedge g)$$

for all $f, g \in L^X$, that is, $\mathcal{N}(x) \wedge \mathcal{N}(y)$ exists and this contradicts the condition (1). Hence, $x \neq y$ implies $\mathcal{M} \not\leq \mathcal{N}(x)$ or $\mathcal{M} \not\leq \mathcal{N}(y)$.

(2) $\Rightarrow$ (3): If (2) holds and $x \neq y$, then $\mathcal{M} \not\leq \mathcal{N}(x)$ or $\mathcal{M} \not\leq \mathcal{N}(y)$ for all ultra fuzzy filters $\mathcal{M}$ on X. From Proposition 1.2, for each fuzzy filter $\mathcal{L}$ on X we find an ultra fuzzy filter $\mathcal{M}$ on X such that $\mathcal{M} \leq \mathcal{L}$ and hence, by means of Lemma 2.2, $\mathcal{L} \not\leq \mathcal{N}(x)$ or $\mathcal{L} \not\leq \mathcal{N}(y)$. Therefore, (3) holds.

(3) $\Rightarrow$ (1): Let $\mathcal{M} \not\leq \mathcal{N}(x)$ or $\mathcal{M} \not\leq \mathcal{N}(y)$ for all fuzzy filters $\mathcal{M}$ on X and for all $x \neq y$ in X. Then there exist $f, g \in L^X$ such that

$$\mathcal{M}(f) \not\geq \mathcal{N}(x)(f) \text{ or } \mathcal{M}(g) \not\geq \mathcal{N}(y)(g)$$

taking $\alpha \in L_0$ for which $\alpha \leq \mathcal{N}(x)(f)$ and $\alpha \leq \mathcal{N}(y)(g)$, we get

$$\alpha \not\leq \mathcal{M}(f) \wedge \mathcal{M}(g) \leq \sup{(f \wedge g)}.$$

Hence, $\mathcal{N}(x) \wedge \mathcal{N}(y)$ does not exist for all $x, y \in X$ with $x \neq y$. $\square$

For a fuzzy topological space (X, τ), a fuzzy filter $\mathcal{M}$ on X is said to converge to a point $x \in X$, written $\mathcal{M} \underset{\tau}{\rightarrow} x$, provided $\mathcal{M}$ is finer than the fuzzy neighborhood filter $\mathcal{N}(x)$, that is, $\mathcal{M} \leq \mathcal{N}(x)$. The conditions (2) and (3) in Theorem 4.1 state that $\mathcal{M} \underset{\tau}{\rightarrow} x$ and $\mathcal{M} \underset{\tau}{\rightarrow} y$ for some fuzzy filters $\mathcal{M}$ on X imply $x = y$.

Example 4.2 Let L be a complete chain, $X = \{x, y\}$ and let $\tau = \{\overline{0}, \overline{1}, x_1, y_1\}$. Then for all $x \neq y$ we find $f = x_1, g = y_1$ such that

$$\sup(f \wedge g) \;=\; 0 < 1 \;=\; \operatorname{int}_\tau f(x) \wedge \operatorname{int}_\tau g(y) \;=\; \mathcal{N}(x)(f) \wedge \mathcal{N}(y)(g)$$

that is, $\mathcal{N}(x) \wedge \mathcal{N}(y)$ does not exist. Hence, (X, τ) is T_2.

Example 4.3 Let L be a complete chain, X a non-empty set and let $\tau = \{\overline{\alpha} \mid \alpha \in L\}$. Then for all $f, g \in L^X$ and $x \neq y$ we have $\operatorname{int}_\tau f = \overline{\inf f}$ and $\operatorname{int}_\tau g = \overline{\inf g}$ and thus

$$\mathcal{N}(x)(f) \wedge \mathcal{N}(y)(g) = \inf f \wedge \inf g \leq \sup(f \wedge g).$$

Hence, $\mathcal{N}(x) \wedge \mathcal{N}(y)$ exists and this means that (X, τ) is not T_2.

A topological space (X,T) is called T_2 if $x \neq y$ implies there are neighborhoods $\mathcal{O}_x$ and $\mathcal{O}_y$ of x and y, respectively such that $\mathcal{O}_x \cap \mathcal{O}_y = \emptyset$ for all $x, y \in X$.

Proposition 4.2 *A topological space (X,T) is T_2 if and only if the induced fuzzy topological space $(X, \omega(T))$ is T_2.*

Proof. (X,T) is T_2 and $x \neq y$ imply there are $\mathcal{O}_x, \mathcal{O}_y \in T$ such that $\mathcal{O}_x \cap \mathcal{O}_y = \emptyset$. Taking $f = \chi_{\mathcal{O}_x}$ and $g = \chi_{\mathcal{O}_y}$, we get

$$\mathcal{N}(x)(f) \wedge \mathcal{N}(y)(g) = (\mathrm{int}_{\omega(T)} f)(x) \wedge (\mathrm{int}_{\omega(T)} g)(y) = 1 > \sup(f \wedge g).$$

Hence, $\mathcal{N}(x) \wedge \mathcal{N}(y)$ does not exist. That is, $(X, \omega(T))$ is T_2.

Conversely, let $(X, \omega(T))$ be T_2 and let $x \neq y$. Then there are $f, g \in L^X$ such that $(\mathrm{int}_{\omega(T)} f)(x) \wedge (\mathrm{int}_{\omega(T)} g)(y) > \sup(f \wedge g)$ and hence $(\mathrm{int}_{\omega(T)} f)(x) > \sup(f \wedge g)$ and $(\mathrm{int}_{\omega(T)} g)(y) > \sup(f \wedge g)$. If we take $\alpha = \sup(f \wedge g)$, then $x \in s_\alpha(\mathrm{int}_{\omega(T)} f)$ and $y \in s_\alpha(\mathrm{int}_{\omega(T)} g)$. Since $\mathrm{int}_{\omega(T)} f$ and $\mathrm{int}_{\omega(T)} g$ are elements of $\omega(T)$, it follows that $s_\alpha(\mathrm{int}_{\omega(T)} f), s_\alpha(\mathrm{int}_{\omega(T)} g) \in T$ hold and hence $\mathcal{O}_x = s_\alpha(\mathrm{int}_{\omega(T)} f)$ and $\mathcal{O}_y = s_\alpha(\mathrm{int}_{\omega(T)} g)$ are neighborhoods of x and y, respectively and moreover $\mathcal{O}_x \cap \mathcal{O}_y = \emptyset$. Hence, (X,T) is T_2. $\Box$

Proposition 4.3 *Let (X,τ) be a fuzzy topological space. Then the following statements*

(1) *(X,τ) is T_2;*

(2) *(X,τ_α) is T_2, $\alpha \in L_1$;*

(3) *$(X, \iota(\tau))$ is T_2*

fulfill the following implications: $(1) \Rightarrow (2) \Rightarrow (3)$.

Proof. $(1) \Rightarrow (2)$: (X,τ) is T_2 and $x \neq y$ imply there are $f, g \in L^X$ such that $\mathrm{int}_\tau f(x) \wedge \mathrm{int}_\tau g(y) > \sup(f \wedge g)$. Taking $\alpha = \sup(f \wedge g)$, then $x \in s_\alpha(\mathrm{int}_\tau f)$ and

$y \in s_\alpha(\mathrm{int}_\tau g)$. Since $(\mathrm{int}_\tau f)(x) \wedge (\mathrm{int}_\tau g)(y) > \alpha$ for all $x \neq y$ in X, then

$$s_\alpha(\mathrm{int}_\tau f) \cap s_\alpha(\mathrm{int}_\tau g) = s_\alpha(\mathrm{int}_\tau f \wedge \mathrm{int}_\tau g) = \emptyset.$$

Because of that $s_\alpha(\mathrm{int}_\tau f), s_\alpha(\mathrm{int}_\tau g) \in \tau_\alpha$ hold, it follows there are neighborhoods $\mathcal{O}_x = s_\alpha(\mathrm{int}_\tau f)$ and $\mathcal{O}_y = s_\alpha(\mathrm{int}_\tau g)$ of x and y, respectively for which $\mathcal{O}_x \cap \mathcal{O}_y = \emptyset$. Therefore, (X, τ_α), $\alpha = \sup(f \wedge g) \in L_1$, $f, g \in L^X$, is T_2.

(2) $\Rightarrow$ (3): Follows directly from Remark 2.1. $\square$

As in the case of T_0-spaces and T_1-spaces we shall show in the following propositions that the initial fuzzy topological space (X, τ) of a family $((X_i, \tau_i))_{i \in I}$ of T_2-spaces is also T_2.

At first Let I be a singleton.

Proposition 4.4 *Let (Y, σ) be a T_2-space and let $f : X \to Y$ be an injective mapping. Then the initial fuzzy topological space $(X, f^{-1}(\sigma))$ is also T_2.*

Proof. Because of that f is injective, then $x \neq y$ in X implies $f(x) \neq f(y)$ in Y and since (Y, σ) is T_2-space it follows there exist $g, h \in L^Y$ such that

$$(\mathrm{int}_\sigma g)(f(x)) \wedge (\mathrm{int}_\sigma h)(f(y)) > \sup(g \wedge h).$$

From the continuity of $f : (X, f^{-1}(\sigma)) \to (Y, \sigma)$ it follows $(\mathrm{int}_\sigma g) \circ f \leq \mathrm{int}_{f^{-1}(\sigma)}(g \circ f)$ for all $g \in L^Y$ and since $\sup(g \wedge h) \geq \sup((g \circ f) \wedge (h \circ f))$ we get

$$(\mathrm{int}_{f^{-1}(\sigma)}(g \circ f))(x) \wedge (\mathrm{int}_{f^{-1}(\sigma)}(h \circ f))(y) > \sup((g \circ f) \wedge (h \circ f)).$$

Thus there exist $k = g \circ f, l = h \circ f \in L^X$ such that

$$(\mathrm{int}_{f^{-1}(\sigma)} k)(x) \wedge (\mathrm{int}_{f^{-1}(\sigma)} l)(y) > \sup(k \wedge l).$$

Hence $(X, f^{-1}(\sigma))$ is T_2-space. $\square$

For any class I we have the following result.

Proposition 4.5 *Let (X_i, τ_i) be a T_2-space for all $i \in I$ and let $f_i : X \to X_i$ be an injective mapping for some $i \in I$. Then the initial fuzzy topological space (X, τ) is also T_2.*

Proof. Since for some $i \in I$, f_i is injective, then $x \neq y$ in X implies $f_i(x) \neq f_i(y)$ in X_i and thus there are $\lambda_i,\ \mu_i \in L^{X_i}$ such that

$$(\mathrm{int}_{\tau_i}\lambda_i)(f_i(x)) \wedge (\mathrm{int}_{\tau_i}\mu_i)(f_i(y)) > \sup(\lambda_i \wedge \mu_i).$$

Since f_i is fuzzy continuous, then $(\mathrm{int}_{\tau_i}\lambda_i) \circ f_i \leq \mathrm{int}_\tau(\lambda_i \circ f_i)$ for all $\lambda_i \in L^{X_i}$. Hence

$$\mathrm{int}_\tau(\lambda_i \circ f_i)(x) \wedge \mathrm{int}_\tau(\mu_i \circ f_i)(y) > \sup(\lambda_i \wedge \mu_i) \geq \sup((\lambda_i \circ f_i) \wedge (\mu_i \circ f_i)).$$

Therefore there exist $\lambda = \lambda_i \circ f_i \in L^X,\ \mu = \mu_i \circ f_i \in L^X$ such that

$$(\mathrm{int}_\tau\lambda)(x) \wedge (\mathrm{int}_\tau\mu)(y) > \sup(\lambda \wedge \mu).$$

Hence, the fuzzy topological space (X, τ) is T_2. $\square$

The following result is a direct consequence of Propositions 4.4 and 4.5.

Corollary 4.1 The fuzzy topological subspace and the fuzzy topological product space of T_2-spaces are also T_2.

In the following it will be shown that the final fuzzy topological space (X, τ) of a family $((X_i, \tau_i))_{i \in I}$ of T_2-spaces is also T_2.

Proposition 4.6 *If (X, τ) is a T_2-space and $f : X \to Y$ a surjective fuzzy open mapping, then the final fuzzy topological space $(Y, f(\tau))$ is also T_2.*

Proof. Since f is surjective, then $y_1 \neq y_2$ in Y implies there are $x_1, x_2 \in X$ such that $y_1 = f(x_1)$, $y_2 = f(x_2)$ and $x_1 \neq x_2$. Because of that (X, τ) is T_2 it follows there are $g, h \in L^X$ such that

$$(\mathrm{int}_\tau g)(x_1) \wedge (\mathrm{int}_\tau h)(x_2) > \sup(g \wedge h)$$

and this means

$$(\mathrm{int}_\tau g)(f^{-1}(y_1)) \wedge (\mathrm{int}_\tau h)(f^{-1}(y_2)) > \sup(g \wedge h)$$

which means

$$(f(\mathrm{int}_\tau g))(y_1) \wedge (f(\mathrm{int}_\tau h))(y_2) > \sup(g \wedge h).$$

Since f is fuzzy open, it follows $f(\mathrm{int}_\tau g) \leq \mathrm{int}_{f(\tau)}(f(g))$ for all $g \in L^X$ and therefore

$$(\mathrm{int}_{f(\tau)} f(g))(y_1) \wedge (\mathrm{int}_{f(\tau)} f(h))(y_2) > \sup(g \wedge h) \geq \sup(f(g) \wedge f(h)).$$

Since $f(g), f(h) \in L^Y$, then we get that the final fuzzy topological space $(Y, f(\tau))$ is T_2. $\square$

Proposition 4.7 *Let I be any class and (X_i, τ_i) be a T_2-space for all $i \in I$ and $f_i : X_i \to X$ be a surjective fuzzy open mapping for some $i \in I$. Then the final fuzzy topological space (X, τ) is also T_2.*

Proof. Since for some $i \in I$, f_i is surjective, then $x \neq y$ in X implies there are $x_i, y_i \in X_i$ such that $x = f_i(x_i)$, $y = f_i(y_i)$ and $x_i \neq y_i$ and thus there are $\lambda_i,\ \mu_i \in L^{X_i}$ such that

$$(\mathrm{int}_{\tau_i} \lambda_i)(x_i) \wedge (\mathrm{int}_{\tau_i} \mu_i)(y_i) > \sup(\lambda_i \wedge \mu_i)$$

and this means

$$(\mathrm{int}_{\tau_i} \lambda_i)(f_i^{-1}(x)) \wedge (\mathrm{int}_{\tau_i} \mu_i)(f_i^{-1}(y)) > \sup(\lambda_i \wedge \mu_i)$$

which means

$$(f_i(\mathrm{int}_{\tau_i} \lambda_i))(x) \wedge (f_i(\mathrm{int}_{\tau_i} \mu_i))(y) > \sup(\lambda_i \wedge \mu_i).$$

Since f_i is fuzzy open, it follows $f_i(\mathrm{int}_{\tau_i} \lambda_i) \leq \mathrm{int}_\tau(f_i(\lambda_i))$ for all $\lambda_i \in L^{X_i}$ and therefore

$$(\mathrm{int}_\tau f_i(\lambda_i))(x) \wedge (\mathrm{int}_\tau f_i(\mu_i))(y) > \sup(\lambda_i \wedge \mu_i) \geq \sup(f_i(\lambda_i) \wedge f_i(\mu_i)).$$

Since $f_i(\lambda_i), f_i(\mu_i) \in L^X$, then we get that the final fuzzy topological space (X, τ) is T_2. $\square$

The following result is a direct consequence of Propositions 4.6 and 4.7.

Corollary 4.2 The fuzzy topological sum space and the fuzzy topological quotient space of T_2-spaces are also T_2.

In the following it will be shown that the finer fuzzy topological space of T_2-space is also T_2.

Proposition 4.8 *Let (X, τ) be a T_2-space and let σ be a fuzzy topology on X finer than τ. Then (X, σ) is also T_2-space.*

Proof. It is easily seen from the properties of the finer topologies. $\square$

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